unknowns U_j, $j = 1,2,...,k$. The resulting system is then solved by PIS, which is believed to have quadratic rate of convergence.

IV. Numerical Results and Discussion

Using DIS and PIS numerical solutions have been computed for a thin symmetric parabolic arc profile and an NACA 0012 profile. The approximate solution Eq. (5b) may also be taken as an initial guess at the solution for the perturbed iterative scheme. Moreover, solutions were also computed with starting solutions other than Eq. (5b), such as

$$u_0(x) = 2[1 - \{1 - u_L(x)\}^{1/2}] \quad (8a)$$

$$u_0(x) = u_L(x) \quad (8b)$$

and

$$u_0(x) = 1 \quad (8c)$$

It is found from computations for the case of a parabolic arc profile, that the flow is entirely subsonic for values of the reduced thickness ratio $\tau < 0.64$, while it is mixed subsonic supersonic for $\tau > 0.64$. In the subsonic range, both PIS and DIS converge very rapidly, requiring less than 10 iteration steps, for results correct to two decimal places. Of these two methods, PIS converges in almost one-half or one-third the number of steps compared to DIS, to the same solution irrespective of the starting solutions Eq. (5b) or Eqs. (8a-c). Further, the super-critical continuous (i.e., shock-free) solution is found in the range

$$0.64 \leq \tau < 0.68 \quad (9)$$

by both methods and they converge to the same super-critical solution, irrespective of the starting solution. Further, computations carried out with 40 and 80 pivotal points for the same profile shape with the same reduced thickness ratio show that there is not mentionable difference in the converged solution for the two choices of the number of pivotal points. However, beyond the range of Eq. (9) there exists another range

$$0.68 < \tau < 0.74 \quad (10)$$

where DIS converges to a continuous (i.e., shock-free) supercritical solution, with any of the starting solutions Eq. (5b) or Eqs. (8a-c). Contrary to expectations, PIS fails to converge in this range. For values of τ greater than 0.74, an expansion shock seems to be formed at the accelerating sonic point and, ultimately, the whole computation diverges.

To test if any solution with shock discontinuity exists, we take a discontinuous solution as the starting solution for values of τ in the range of Eq. (9). However, PIS does not converge with such starting solutions, whereas DIS converges to a smooth solution shown in Fig. 1. The converged solution, correct to two decimal places obtained by DIS, for a parabolic arc profile with reduced thickness ratio $\tau = 0.74$, is shown in Fig. 2. Only 12 iteration steps are needed for convergence. For $\tau = 0.66$, with 20 pivotal points, a peculiar feature is to be seen when PIS converges in 5 steps to a slightly asymmetric flow, shown in Fig. 3. This asymmetry is ascribed to the larger truncation error in the case of a smaller number of pivotal points.

The number of iteration steps needed for convergence, correct to two decimal places, for a parabolic arc profile with 40 pivotal points is shown in Table 1, which clearly shows the faster rate of convergence of PIS.

In the case of an NACA 0012 profile also, a range of freestream Mach numbers have been found where both the schemes converge to a shock-free supercritical solution.

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H-R_x Method for Predicting Transition

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Introduction

THIS Note gives a brief description of a shortcut method, the H-R_x method, for predicting transition in a wide class of boundary-layer flows, including the effects of pressure gradient, surface heat transfer, and suction. Here H and R_x are the body shape factor and the Reynolds number, respectively, based on distance x measured in the direction of the flow. The method is extremely simple to use and a good substitute to the well known but rather complicated e^9 method.

The e^n Method of Forecasting Transition

The H-R_x method has not been correlated with test data but rather has been justified in terms of Tollmien-Schlichting waves and e^9 type calculations. Data are gradually being accumulated that show that the e^9 method is the best all-around method that now exists for predicting boundary-layer transition. Since the H-R_x method is rooted in the e^9 method, we proceed to make some comments about the latter.

Transition, although it may commence with the amplification of Tollmien-Schlichting waves as described by linear stability theory, is dominated in its late stages by three dimensional and nonlinear effects. Why then does transition

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occur at a disturbance amplification ratio a_t of about $a_t \approx e^9$ (Ref. 1) as computed from linear instability theory?

The e^9 method is rooted in the following observations: Liepmann[2] hypothesized that at the breakdown to turbulence, the Reynolds stress $\tau = -\rho\overline{u'v'}$ due to the amplified fluctuations becomes comparable in magnitude to the maximum mean laminar shear stress $\tau_L = \mu(\partial\bar{u}/\partial y)$. The ratio τ/τ_L is given by

$$\tau/\tau_L = \frac{2}{c_{f_L}}\left[kb\left(\frac{u_n'}{u_e}\right)^2 a^2(x)\right]_{\max} \tag{1}$$

where c_{f_L} is the skin friction coefficient, u_n' is the disturbance amplitude at the neutral point, $b = v'/u'$ and $k = \overline{u'v'}/uv$, $a(x)$ is the amplification factor with respect to the neutral point, u and v are velocities in the x and y direction, respectively, and prime indicates disturbance values. Obremski et al.[3] observe that in a low turbulence environment the critical Tollmien-Schlichting mode at the beginning of amplification may possess an amplitude u_n' of the order of 0.001% u_e, or perhaps even less, where u_e is the edge velocity. Furthermore, Klebanoff et al.[4] found, in detailed examination of flat plate measurements, that disturbance growth via the laminar instability mechanism ceases to be valid when the rms velocity fluctuation u' in the boundary layer reaches $(u'/u_e)_{\max} \approx 0.015$, but that the first appearance of turbulent spots is expected at about $(u'/u_e)_{\max} \approx 0.20$. Assuming that these figures represent disturbance growth *not* only on a flat plate but *also* on airfoils and bodies of revolution, all at low freestream turbulence level, we find that amplification in the linear regime $[(u'/u_e)_{\max}/(u_n'/u_e) \approx 0.015/0.00001 = 1500 \approx e^{7.35}]$ substantially exceeds the amplification in the nonlinear regime $[0.20/0.015 \approx 13 \approx e^{2.6}]$. The total amplification (=0.20/0.00001) at transition is seen to be about $20{,}000 \approx e^{9.9}$; this amplification is of the same order as that reported by Michel in Ref. 5, $e^{9.2}$, and computed by Smith et al.[6] and van Ingen[7] from linear instability theory. This amplification factor is consistent with the hypothesis of Liepmann. If we set $\tau/\tau_L = 1$, $(u_n'/u_e) \approx 0.00001$, $c_{f_L} \approx 0.664 R_L^{-1/2}$ with $R_L = 3 \times 10^6$ (for transition on a flat plate), $(u'/v')_{tr} = 0(10)$ and $(uv/\overline{u'v'})_{tr} = 0(10)$ at transition, then Eq. (1) gives an amplification at transition $a(x_{tr}) \approx 1.4 \times 10^4 = e^{9.5}$. This suggests that in boundary-layer flows, where freestream turbulence is very low, the growth of Tollmien-Schlichting waves controls the development to turbulent flow and so linear instability theory can be used as a basis for forecasting transition.

Some Predictions from Stability Theory and the e^9 Method

The factors being considered in this Note, for either two-dimensional or axisymmetric low-speed flow, are the effects of 1) pressure gradient, 2) suction, and 3) wall heating or cooling. Figure 1 shows the results of calculations of the neutral stability point for wedge and other flows under a large variety of conditions. It contains results for an adiabatic flat plate with varying degrees of mass transfer, including the asymptotic suction case. The largest number of points are for various wedge flows in water with and without heat. The significant fact is that when the neutral stability point is plotted in the form of the critical Reynolds number $R_{\delta^*\mathrm{crit}}$ vs the shape factor H, this rather considerable variety of flows forms a *single* well defined curve. Shape factor is the immediate determinant of neutral stability rather than some more remote measure such as Hartree's β or Pohlhausen's Λ.

The present authors proceeded to determine if a similar correlation existed at transition as predicted by the e^9 method. Suction was not included in the study but a considerable number of wedge flows with various wall temperatures were studied for water. The results are shown in Fig.

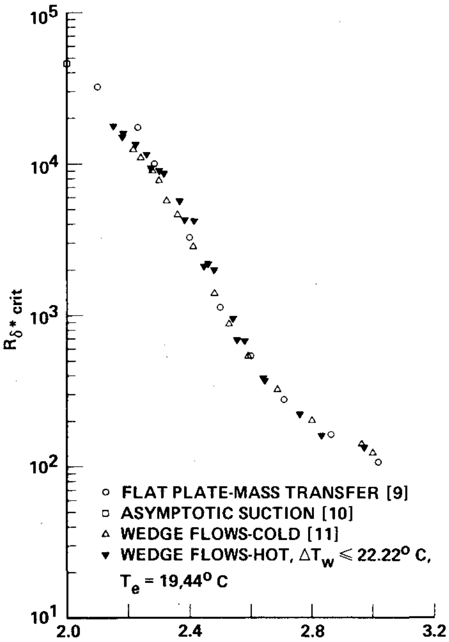

Fig. 1 Critical Reynolds number $R_{\delta^*\mathrm{crit}}$ vs H.

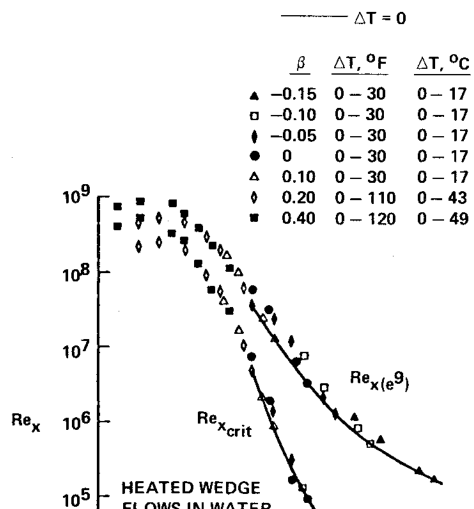

Fig. 2 Critical and transition (computed) Reynolds number for adiabatic and heated wedge flows in water. β is Hartree's β.

2, which also includes the data of Fig. 1. But in Fig. 2 the calculated values are plotted in terms of R_x vs H rather than R_{δ^*} vs H. Again, except for the highest temperature differentials the data form well defined curves; the scatter is so small that suction effects would undoubtedly fall along the same curve. The correlation indicates that the neutral and transition points are primarily a function of H. Pressure gradient, heating, and suction are only methods of influencing H. The results of Fig. 2 are only for two-dimensional and similar flows, but they suggest a method of predicting transition: for the conditions of the problem, make a plot of H vs R_x, from ordinary boundary-layer calculations, as one proceeds back along the body. The plot will always start beneath the two loci of Fig. 2. When the curve

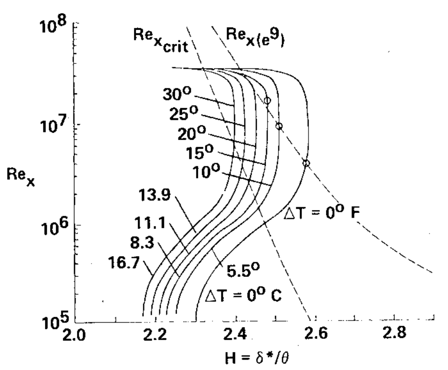

Fig. 3 Paths of boundary-layer development (H vs R_x) and predicted transition for a 13:1 Reichardt body. Comparison with heated wedge-flow predictions. Circles denote point where $Re_{s(e^9)}$ is reached, $u_\infty/\nu = 20\times10^6$/m ($6\times10^6$/ft). s is length measured along the body surface.

crosses the e^9 locus, transition should occur. An equation that fits the e^9 locus very well is (for bodies of revolution R_x is replaced with R_s where s is the distance measured along the surface of the body):

$$\log[R_x(e^9)] = -40.4557+64.8066H-26.7538H^2$$
$$+3.3819H^3, \quad 2.1<H<2.8 \tag{2}$$

H-R_x Method for Predicting Transition over a Body of Revolution

Because the correlationsof Figs. 1 and 2 are for two-dimensional similar flows, as a sample problem we chose a heated Reichardt body of revolution moving in 19.44°C (67°F) water at a velocity that gives $u_\infty/\nu = 20\times10^6$/m ($6\times10^6$ ft) (Fig. 3). Because there is a stagnation point the flow starts out very stable (low H). For the case of no heating ($\Delta T = 0$°C) the neutral point is crossed at $R_x \approx 10^6$ and transition occurs between $R_x = 3$ and 4×10^6. The circle is for the full e^9 calculation, so the agreement is perfect. With 5.5°C (10°F) of heating, the boundary layer is more stable and now the neutral point is crossed at about $R_x = 1.2\times10^6$ and transition is moved back to about $R_x = 9\times10^6$. Again the agreement with the full e^9 method is very good. At a heating of 11.1°C (20°F) transition is moved still further back but the agreement with the full e^9 calculation is not quite as good. But the example shows clearly the method of prediction and that the correlation seems to work even when the flow is nonsimilar and non-two-dimensional.

For a practical shape such as in this example, the H-R_x trace meanders. For a pure similar flow the trace would be vertical. If there is a sudden change in pressure gradient, suction, or heat the trace would be horizontal for some distance. Note that only ordinary but accurate boundary-layer calculations of H vs R_x are needed for transition prediction, not lengthy stability calculations.

Discussion

Other simple methods of predicting transition lack the generality of the H-R_x method because they are correlations based on parameters that are more removed from the best measure of stability H. Michel's method for instance is an excellent correlation method, correlating R_θ vs R_x at transition for two-dimensional imcompressible flow, e.g., airfoils. But when bodies of revolution, heat, or suction are considered the correlation is no longer applicable. The same kind of trouble applies to methods like Granville's,[8] which correlates with a pressure gradient parameter such as Pohlhausen's Λ. Obviously this kind of method fails if other means than pressure gradients are used to affect transition. Consider a fixed point on a particular body. That point has one value of Λ regardless of the degree of suction or heating. But if H is used as the correlating parameter, changes at that point in the boundary-layer flow, as by heating or suction, manifest themselves in changes in H. However, we must stress that when there is heat transfer more than the shape factor is involved, particularly if temperature differences are high. For instance, in the full e^9 method using the extended Orr-Sommerfeld Eq., Ref. 1, additional terms such as $d^2\mu/dy^2$ enter the stability problem in addition to the basic velocity profile data that determines H. Note that the correlation is poor at high heating rates where $H<2.1$, Fig. 2.

The H-R_x method for predicting transition gives reasonable answers so long as the flow does not vary too much from nearly similar flow, i.e., local similarity. Also it is applicable only so long as the effects of surface roughness, vibrations, and freestream turbulence level are sufficiently low, just as required for the basic e^9 method. The method is also restricted to heating rates where $T_w - T_\infty$ does not exceed about 23°C.

The correlation has been developed entirely theoretically using the e^9 method as if it were exact for predicting transition. No attempt has been made to develop the correlation from experimental data, partly because of the labor involved but partly because good test data are quite scarce for unusual conditions such as heating in water. The main fact that can be claimed therefore is that the H-R_x method is a very convenient substitute for calculating nearly the same results as the full e^9 method. Also of course this sample case is not the only one that has been studied; the H-R_x method has been used extensively in other studies with plausible predictions, and a number of other comparisons with the full e^9 method have been made. (These studies are bottled up in the classified literature.) Where the flow differs widely from the conditions of the correlation, a check should be made by the full e^9 method. But for a large range of practical conditions, the H-R_x method seems accurate, convenient and quite general. In Fig. 1, the data for a flat plate with mass transfer are from Tsou and Sparrow,[9] the data for the asymptotic suction profile are from Hughes and Reid,[10] and the unheated wedge flow data are from Wazzan et al.[11]

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